

1. Affine algebraic sets

1.1. Recollection on commutative algebra:

In this class, all rings are commutative with a unit element for multiplication!

Let R be a ring.

• $a \in R$ is a unit if $\exists b \in R$ s.t. $a \cdot b = 1$

We set $R^\times = \{\text{units in } R\}$.

• $a \in R$ is a zero-divisor if $\exists b \in R \setminus \{0\}$ s.t. $a \cdot b = 0$.

• R is an integral domain, or just domain, if $\forall a, b \in R, a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$ (i.e. R has no zero-divisors).

• R is reduced if it contains no nilpotents i.e. $a^n = 0$ for some $n \geq 1 \Rightarrow a = 0$.

• Any domain R has a quotient field $Q(R) = K$

As a set $K = \left\{ \frac{a}{b} \mid a, b \in R, b \neq 0 \right\} / \left\{ \frac{a}{b} \sim \frac{c}{d} \Leftrightarrow ad = bc \right\}$.

A subset $I \subset R$ is an ideal if $\forall x \in R, a, b \in I$
 $x \cdot a \in I, a + b \in I$.

In that case the quotient R/I is again a ring.

• $I \subset R$ is a proper ideal if $I \neq R$

• A proper ideal I is maximal if it's not contained in any larger proper ideal

• A proper ideal I is prime if $\forall a, b \in R$
 $a \cdot b \in I \Rightarrow a \in I$ or $b \in I$.

• $I \subset R$ is radical if $f^n \in I \Rightarrow f \in I \quad \forall f \in R, n > 0$.

• For any $I \subset R$, the radical $\text{Rad}(I)$ (or $\sqrt{I}$) is

- For any $I \subset R$, the radical $\text{Rad}(I)$ (or $\sqrt{I}$) is
 $\text{Rad}(I) = \{a \in R \mid a^n \in I \text{ for some } n > 0\}$.

Lemma: $I \subset R$ is

maximal $\Leftrightarrow R/I$ is a field

prime $\Leftrightarrow R/I$ is a domain.

radical $\Leftrightarrow R/I$ is reduced

We have maximal $\Rightarrow$ prime $\Rightarrow$ radical.

Exple: $(x) \subset k[x, y]$ is prime but not maximal

$(xy) \subset R$ is radical $\equiv$ prime

Given a subset $S \subset R$, we can consider the smallest ideal containing S :

$$I(S) = \left\{ \sum_{i=1}^n a_i s_i \mid s_i \in S, a_i \in R, n \in \mathbb{N} \right\}$$

I is finitely generated if $I = I(S)$ for

some finite set $S = \{s_1, \dots, s_n\}$.

In this case we write $I = (s_1, \dots, s_n)$.

- R is Noetherian if $\nexists$ infinite chain of ideals $I_1 \subsetneq I_2 \subsetneq I_3 \dots$
 $\Leftrightarrow$ Every ideal is finitely generated.

Hilbert's basis thm: R Noeth $\Rightarrow R[x]$ Noeth.

In part. $k[x_1, \dots, x_n]$ is Noeth. for any field k .

- I is principal if it's generated by one element.

A domain R is a principal ideal domain (PID), if every ideal is principal.

Exples: $R = \mathbb{Z}$. $I = (5)$ is principal and maximal

$\hookrightarrow \mathbb{Z}/5\mathbb{Z}$ is a field.

$I = (6)$ is principal, but not prime

$\hookrightarrow \mathbb{Z}/5\mathbb{Z}$ is a field.

• $I = (6)$ is principal, but not prime
 $2 \cdot 3 \in (6)$ but $2, 3 \notin (6)$.

• $I = (a_1, \dots, a_n) = (\gcd(a_1, \dots, a_n)) \leadsto \mathbb{Z}$ is a PID.

• Similarly $K[X]$ is a PID, but not $K[X_1, \dots, X_n] \quad \forall n \geq 2$.

• $a \in R$ is irreducible if a is not a unit $\neq 0$
and if $a = b \cdot c$ with $b, c \in R$ then $b \in R^\times$ or $c \in R^\times$.

• R is a unique factorization domain (UFD)

if R is a domain and elements in R can be factored uniquely, up to units and reordering, into irreducible elements.

R UFD, a principal ideal $(a) \subset R$ is prime $\Leftrightarrow a$ irred.

Exple: $R = \mathbb{Z}$, $a \in \mathbb{Z}$ irred. $\Rightarrow a$ or $-a$ is prime

• R a UFD $\Rightarrow R[X]$ a UFD

By induction $R[X_1, \dots, X_n] = R[X_1, \dots, X_{n-1}][X_n]$

is a UFD.

• Every PID is a UFD

Gauss Lemma: R a UFD and $a \in R[X]$ non-const., irred.,

$a \in Q(R)[X]$ is still irreducible.

Localization: Let R be a domain.

• If $S \subset R$ is a mult. subset (i.e. $1 \in S, a, b \in S \Rightarrow a \cdot b \in S$)

The localization $S^{-1}R = \{x \in Q(R) \mid x = \frac{a}{b} \text{ w. } b \in S\}$

If M is an R -module we have similarly

$$S^{-1}M = \left\{ \frac{m}{s} \mid m \in M, s \in S \right\} / \left\{ \frac{m}{s} = \frac{m'}{s'} \Leftrightarrow m s' = m' s \right\}$$

• If $\mathfrak{p} \subset R$ is a prime ideal, $R \setminus \mathfrak{p}$ is mult. and we define

$$R_{\mathfrak{p}} := (R \setminus \mathfrak{p})^{-1}R = \left\{ x \in Q(R) \mid x = \frac{a}{b} \text{ w. } b \notin \mathfrak{p} \right\}.$$

$$M_{\mathfrak{p}} := (R \setminus \mathfrak{p})^{-1}M$$

$$M_p := (R_p)^{\sim} M.$$

and $M_p := (R_p)^{\sim} M$.

There is a 1-1 corresp. $\{\text{Ideals } \subset \mathfrak{p}\} \leftrightarrow \{\text{Ideals in } R_p\}$
 i.p. R_p is a local ring. $I \mapsto I_p$

Localization is exact, i.p. if $I \subset \mathfrak{p}$ the s.a.s. of R -modules

$$0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$$

Becomes

$$0 \rightarrow I_p \rightarrow R_p \rightarrow (R/I)_p \rightarrow 0$$

$$\rightarrow (R/I)_{p/I} \simeq R_p / I_p$$

$$\underline{\text{Expl:}} \quad S = R \setminus \{0\} \leadsto S^{-1}R = \mathbb{Q}(R) = R_{(0)}$$

$$R = k[x_1, x_2], I = (x_1, x_2), \text{ then } R_I = \{f \in \mathbb{Q}(R) \mid f = \frac{a}{b}, b(0, \dots, 0) \neq 0\}$$

$\hookrightarrow$ "fct's at 0".

Polynomial rings: For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we set $X^\alpha := X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n} \in k[x_1, \dots, x_n]$

Thus any $F \in k[x_1, \dots, x_n]$ can be written as $F = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha X^\alpha$. ($\mathbb{N} = \{0, 1, \dots\}$)

F is homogeneous or a form of degree d if $\lambda_\alpha = 0$ unless $\alpha_1 + \dots + \alpha_n = d$

Any F can be written uniquely as $F = F_0 + \dots + F_d$ w/ F_i a form of

The integer d is called the degree of F . degree i , $F_i \neq 0$

The derivative of $F = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha X^\alpha$ with respect to x_i is

$$\bar{F}_{x_i} = \frac{\partial F}{\partial x_i} = \sum_{\alpha \in \mathbb{N}^n} a_i \lambda_\alpha X_1^{\alpha_1} \dots X_{i-1}^{\alpha_{i-1}} X_i^{\alpha_i-1} X_{i+1}^{\alpha_{i+1}} \dots X_n^{\alpha_n}.$$

$$T_{X_i} = \frac{\partial}{\partial x_i} = \sum_{a \in \mathbb{N}^n} a_i \lambda_a X_1^{a_1} \cdots X_n^{a_n}.$$

The usual rules for partial derivatives apply.

If F is a form of degree d we have Euler's Theorem

$$d \cdot F = \sum_{i=1}^n X_i F_{X_i}$$

1.2. Affine space and algebraic sets:

Let k be a field.

Def: For every $n \geq 0$ the affine n -space A_k^n (or just A^n , if k is understood), is the set k^n .

In particular A^0 is a point. A^1 is called the affine line, A^2 the affine plane.

Given a set $S \subset k[x_1, \dots, x_n]$ of polynomials, we set

$$V(S) = \{x = (x_1, \dots, x_n) \in A^n \mid f(x_1, \dots, x_n) = 0 \ \forall f \in S\}.$$

If $S = \{f_1, \dots, f_k\}$ we write $V(f_1, \dots, f_k)$ for $V(S)$.

The 0-set of a single non-const. polynomial $V(f)$ with $f \in k[x_1, \dots, x_n]$ is called a hypersurface.

If $n=2$, then $V(f)$ is an affine plane curve.

A subset $V \subset A^n$ is algebraic if $V = V(S)$ for some $S \subset k[x_1, \dots, x_n]$.

Exmples:

• $S = \emptyset \quad V(S) = A^n$

• $S = k[x_1, \dots, x_n] \quad V(S) = \emptyset.$

• $n=1 \quad f \in k[x], \quad k \text{ alg. closed},$

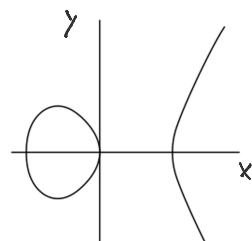
Then $V(f) = \{x \in k \mid f(x) = 0\} = \text{roots of } f.$

• $n=2, \quad k = \mathbb{R}.$

– $f(x, y) = y^2 - x(x^2 - 1) \rightsquigarrow$

affine elliptic curve

$V(f) =$

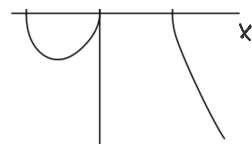
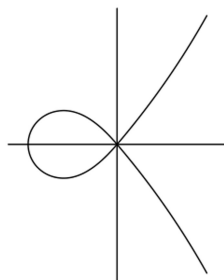


affine elliptic curve

$$-f(x,y) = y^2 - x^2(x+1)$$

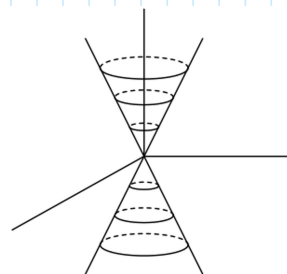
nodal curve

$$V(f) =$$



$$-f(x,y,z) = z^2 - x^2 - y^2$$

$$V(f) =$$



$$-f(x,y) = x^2 + y^2$$

$$V(f) = \{0\} \subset \mathbb{A}_{\mathbb{R}}^2$$

but $V(f) \subset \mathbb{A}_{\mathbb{C}}^2$ is much bigger since $f(x,y) = (x-iy)(x+iy)$.

$\leadsto$ The ground field K is important.

Typically we want it to be algebraically closed (for example $K = \mathbb{C}$), but for visualizations we'll use $K = \mathbb{R}$.

Non-examples: $-\{x \in \mathbb{A}_{\mathbb{R}}^1 \mid x \geq 0\}$

$$-\{x \in \mathbb{A}_{\mathbb{C}}^1 \mid |x| = 1\}$$

Lemma 1.1:

1) Let $S \subset K[x_1, \dots, x_n]$ and I the ideal generated by S . Then $V(S) = V(I)$. I.p. every $V \subset \mathbb{A}^n$ algebraic is of the form $V = V(I)$ for some ideal $I \subset K[x_1, \dots, x_n]$.

2) Let $\{I_\alpha\}$ be a collection of ideals in $K[x_1, \dots, x_n]$. Then $V(\bigcup_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$

$\leadsto$ Intersection of alg. sets is algebraic.

3) If $I \subset J$, then $V(J) \subset V(I)$

4) For polynomials $f, g \in K[x_1, \dots, x_n]$ $V(f) \cup V(g) = V(f \cdot g)$

For ideals $I, J \subset K[x_1, \dots, x_n]$ $V(I) \cup V(J) = V(I \cdot J)$

where $I \cdot J := \{f \cdot g \mid f \in I, g \in J\}$.

5) For $a = (a_1, \dots, a_n) \in \mathbb{A}^n$ $V(\{x_1 - a_1, x_2 - a_2, \dots, x_n - a_n\}) = \{a\}$

5) For $a = (a_1, \dots, a_n) \in A^n$ $V(\{x_1 - a_1, x_2 - a_2, \dots, x_n - a_n\}) = \{a\}$
 Any point in A^n is alg $\leadsto$ Any finite set in A^n is alg.

1) Let $h \in \sum f_i \cdot g_i \in I$ w/ $f_i \in S$ and $x \in V(S)$.

Then $f_i(x) = 0 \ \forall i \leadsto h(x) = 0 \leadsto x \in V(I) \leadsto V(S) \subset V(I)$

If $x \in V(I)$, then i.p. $f(x) = 0 \ \forall f \in S \subset I \leadsto x \in V(S) \leadsto V(S) \supset V(I) \checkmark$

2) $x \in V(\bigcup_{\alpha} I_{\alpha})$. Then for any α and $f \in I_{\alpha}$ $f(x) = 0$
 $\leadsto x \in V(I_{\alpha}) \leadsto x \in \bigcap V(I_{\alpha})$

Conversely $x \in \bigcap V(I_{\alpha})$ and $f \in \bigcup_{\alpha} I_{\alpha}$, then $f \in I_{\alpha}$ for some α .
 $\leadsto f(x) = 0 \leadsto x \in V(\bigcup_{\alpha} I_{\alpha}) \checkmark$

3-5) Exercise □

Since $K[x_1, \dots, x_n]$ is Noetherian (by Hilbert's basis theorem)
 every ideal I is finitely generated i.e. $I = (f_1, \dots, f_k)$
 for some $f_i \in K[x_1, \dots, x_n]$. As a corollary we thus get

Corollary 1.2: Every algebraic $V \subset A^n$ is of the form

$$V = V(f_1, \dots, f_k) = V(f_1) \cap \dots \cap V(f_k) \text{ for some } f_1, \dots, f_k \in K[x_1, \dots, x_n]$$

1.3 Ideal of a set of points and the Nullstellensatz

By the previous lemma we have a map

$$\{\text{Ideals in } K[x_1, \dots, x_n]\} \xrightarrow{V} \{\text{alg sets in } A^n\}$$

Conversely, for any subset $X \subset A^n$ we define

$$I(X) := \{f \in K[x_1, \dots, x_n] \mid f(x) = 0 \ \forall x \in X\} \subset K[x_1, \dots, x_n].$$

Clearly $I(X)$ is an ideal: $f, f' \in I(X)$, $f(x) + f'(x) = 0 \ \forall x \in X \leadsto f + f' \in I(X)$
 $f \in I(X)$, $g \in K[x_1, \dots, x_n]$, $g(x) \cdot f(x) = 0 \ \forall x \in X \leadsto g \cdot f \in I(X)$.

Lemma 1.3: 1) If $X \subset Y$, then $I(X) \supset I(Y)$

2) For $J \subset K[x_1, \dots, x_n]$ an ideal, $I(V(J)) \supset J$

2) For $\mathcal{I} \subset K[x_1, \dots, x_n]$ an ideal, $I(V(\mathcal{I})) \supset \mathcal{I}$

3) For $W \subset \mathbb{A}^n$ alg. $V(I(W)) = W$

pf: 1) $f \in I(Y) \leadsto f$ vanishes on $Y \leadsto f$ vanishes on $X \leadsto f \in I(X)$

2) $I(V(\mathcal{I})) = \{f \in K[x_1, \dots, x_n] \mid f(x) = 0 \ \forall x \in V(\mathcal{I})\} \supset \mathcal{I}$

3) By definition $V(I(X)) \supset X$ for any $X \subset \mathbb{A}^n$. If $X = V(\mathcal{I})$ alg,

then $V(I(X)) = V(I(V(\mathcal{I}))) \subset V(\mathcal{I}) = X$
 $\uparrow$ By Lemma 1.1.3) $\square$

There are essentially two reasons why $I(V(\mathcal{I})) \supset \mathcal{I}$ in general:

1) $\mathcal{I} = (x^n) \subset K[x]$, $\leadsto V(x^n) = \{0\} \leadsto I(\{0\}) = (x) \subset K[x]$

2) $(x^2+1) \in \mathbb{R}[x] \leadsto V(\mathcal{I}) = \emptyset \leadsto I(\emptyset) = \mathbb{R}[x]$.

The first phenomenon is easy to understand, as we have $\text{Rad}(x^n) = (x)$.

Lemma 1.4: For any $X \subset \mathbb{A}^n$, $I(X)$ is a radical ideal.

pf: If $f^n \in I(X)$ for some n , then $f(x)^n = 0 \ \forall x \in X \leadsto f(x) = 0 \ \forall x \in X \leadsto f \in I(X) \ \square$

The second phenomenon comes from the fact, that $\mathbb{R}$ is not alg closed. If we replace $\mathbb{R}$ with $\mathbb{C}$ we'd have

$(x^2+1) \in \mathbb{C}[x] \leadsto V(\mathcal{I}) = \{\pm i\} \leadsto I(\{\pm i\}) = (x^2+1)$.

Thm 1.5 (Hilbert's Nullstellensatz) Let K be algebraically closed

$\mathcal{I} \subset K[x_1, \dots, x_n]$. Then

$$I(V(\mathcal{I})) = \text{Rad}(\mathcal{I}).$$

pf: Rings and Modules

In particular we have a 1-1 correspondence

$$\{\text{Radical ideals in } K[x_1, \dots, x_n]\} \xrightleftharpoons[I]{V} \{\text{Algebraic subsets of } \mathbb{A}^n\}$$

Under this corresp. $\{\text{maximal ideals}\} \leftrightarrow \{\text{points in } \mathbb{A}^n\}$:

Thm 1.6 (Weak Nullstellensatz): If K is algebraically closed

Under this corresp. $\{\text{maximal ideals}\} \rightarrow \text{points in } \mathbb{A}^n :$

Thm 1.6 (Weak NS) : Let K be algebraically closed.

Every maximal ideal $I \subset K[x_1, \dots, x_n]$ is of the form

$$\underline{I} = (X_1 - a_1, \dots, X_n - a_n) \text{ with } a = (a_i) \in A^n.$$

pt: Rings and Modules.

Corollary 1.7: Let $I \subset K[X_1, \dots, X_n]$ be any ideal. Then

$V(\bar{I})$ is a finite set $\Leftrightarrow k[X_1, \dots, X_n]_{\bar{I}}$ is a finite dim.

k -vector space.

In this case $|V(I)| \leq \dim_K k[x_1, \dots, x_n]/I$.

p1: Let $I \subset K[x_1, \dots, x_n]$ be any ideal and $P_1, \dots, P_r \in V(I)$ distinct points. We can choose (Exercise) $F_1, \dots, F_r \in K[x_1, \dots, x_n]$

$$\text{z.B. } F_i(P_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

We write $f_1, \dots, f_r$ for the residues of $F_1, \dots, F_r$ in $k[X_1, X_2]_I$.

Claim: $f_1, \dots, f_r$ are lin. indep.

for $i, j \leq n$

pt: Assume $\sum \lambda_i f_i = 0 \leadsto \sum \lambda_i F_i \in I \leadsto 0 = \sum \lambda_i F_i(P_j)$

$\rightarrow \lambda_j = 0 \quad 1 \leq j \leq r \rightarrow f_i$ are lin indep. v.

It follows that: $\dim_k k[x_1, \dots, x_n]_T < \infty \Rightarrow |V(T)| < \infty$

and in this case $\dim_k k[X_1, \dots, X_n]_{/I} = |V(I)|$.

Now assume $V(I) = \{P_1, \dots, P_r\} \in A^n$, write $P_i = (a_{i1}, \dots, a_{in}) \in A^n$

and define $F_j = \prod_{i \in I} (x_i - d_{ij})$. Then $F_j \in I(V(I)) \stackrel{HNS}{=} \text{Rad}(I)$.

$$\leadsto \exists N > 0 \text{ s.t. } F_d^N \in I \text{ for } 1 \leq j \leq n$$

$\Rightarrow f_i^N = 0$ in $k[x_1, x_2]_{\mathcal{I}}$. But $f_i^N = x_i^{N \cdot r} + \text{lower order terms}$.

$\Rightarrow X_j^{Nr}$ is a K -linear comb. of $\{1, X_j, X_j^2, \dots, X_j^{Nr-1}\}$

$$n: x_i^s \quad // \quad // \quad \text{for any } s \geq 0.$$

taking products for different j 's we see that the finite set

$\{x_1^{m_1} \dots x_n^{m_n} \mid m_i < N-v\}$ generates $K[x_1, \dots, x_n]/I$ as a K -vector space $\square$

Remk: Usually one first proves the weak NS and then uses it to deduce the HNS.

• These two theorems are the reason we assume from now on, that K is algebraically closed without writing it every time

Question we'll answer next week: How to characterize $V(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$?

1.4 Irreducible alg. sets:

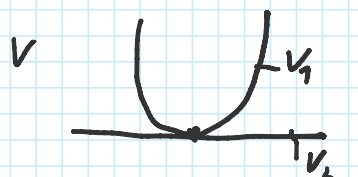
Def: An alg. subset $V \subset A^n$ is irreducible if for all $W_1, W_2 \subset A^n$ alg. w. $W_1 \cup W_2 = V$ either $W_1 = V$ or $W_2 = V$.

• V is reducible if it's not irreducible.

Exmp: $V = \{x_1, \dots, x_n\} \subset A^n$ is a collection of points, then

V is irreducible $\Leftrightarrow K=1$.

• $f(x, y) = y(x^2 - y) \in K[x, y]$, $V = V(f) \subset A^2$

V  not irred. $V = V_1 \cup V_2$
and $V_1 = V(x^2 - y)$, $V_2 = V(y)$.

Proposition 1.8: An algebraic set V is irred. if and only if $I(V)$ is a prime ideal

Pf: If $I(V)$ is not prime, take $f_1, f_2 \notin I(V)$ s.t. $f_1 f_2 \in I(V)$

pf: If $I(V)$ is not prime, take $f_1, f_2 \in I(V)$ s.t. $f_1 \cdot f_2 \in I(V)$

Then $V = (V \cap V(f_1)) \cup (V \cap V(f_2))$, since $\forall x \in V \quad f_1(x) \cdot f_2(x) = 0$
 $\leadsto V$ is reducible

Conversely, if $V = W_1 \cup W_2$ w/ $W_i \subsetneq V$ for $i=1,2$. Then

$I(W_i) \supsetneq I(V)$. Take $f_i \in I(W_i) \setminus I(V)$

$\leadsto f_1 \cdot f_2 \in I(W_1) \cap I(W_2) = I(W_1 \cup W_2) = I(V) \leadsto I(V)$ not prime $\square$

If $V \subset \mathbb{A}^n$ is reducible one can decompose it $V = W_1 \cup W_2$
then do the same with W_1, W_2 etc... This process has to
stop eventually as $k[x_1, \dots, x_n]$ is Noetherian by the
Hilbert Basis thm. This leads to

Thm 1.8: For every $V \subset \mathbb{A}^n$ alg. there exist unique irreducible
alg sets $V_1, \dots, V_k \subset \mathbb{A}^n$ s.t. $V = V_1 \cup V_2 \dots \cup V_k$
and $V_i \not\subset V_j$ for all $i \neq j$.

pf: For a complete proof see rings and modules 7.4.8.

Def: $V_1, \dots, V_k$ are called the irreducible components of V

Remark: Applying I we get $I(V) = I(V_1) \cap \dots \cap I(V_k)$

w/ $I(V_i)$ prime ideals. This was called the minimal primary
decomposition of $I(V)$ in rings and modules.

Exple: $I = (x, y, z) \subset k[x, y, z]$

$$V(I) = \{ (x, y, z) \in \mathbb{A}^3 \mid \begin{matrix} x=0 \\ y, z=0 \end{matrix} \} = V(x, y) \cup V(y, z)$$

Algebraically $I = (x, y) \cap (x, z)$

In general it is quite difficult to find this decomposition, or even
to check whether a given V is irreducible. However for hypersurfaces
it is simpler

it is simpler

• $I = (F) \rightsquigarrow F \in K[X_1, \dots, X_n]$. Let $F = F_1^{d_1} \cdots F_k^{d_k}$ be its factorization into irreducible factors. Then

$$V(F) = V(F_1^{d_1} \cdots F_k^{d_k}) = V(F_1) \cup \dots \cup V(F_k)$$

is the decomp. into irred. components.

1.5 Algebraic subsets of the plane $\mathbb{A}^2$

To get a better feeling of the concepts, let's classify irreducible algebraic subsets of $\mathbb{A}^2$.

Lemma 1.9: Let $F, G \in K[X, Y]$ non-const. with no common factor. Then $V(F, G) = V(F) \cap V(G)$ is a finite set of points.

pt.: By Gauss' Lemma F, G have no common factors in $K(X)[Y]$

But $K(X)[Y]$ is a PID $\rightsquigarrow (F, G) = (1)$ i.e. $A, B \in K(X)$ s.t.

$$A \cdot F + B \cdot G = 1$$

Also $\exists C \in K[X]$ s.t. $AC, B \cdot C \in K[X, Y]$

$\rightsquigarrow CA \cdot F + CB \cdot G = C$ in $K[X, Y]$, but C does not depend on Y .

Let $(x, y) \in V(F, G) \subset \mathbb{A}^2 \rightsquigarrow C(x) = 0$

$\rightsquigarrow$ only finitely many possibilities for x .

Switching X and Y proves the lemma $\square$

Corollary 1.10: The irreducible algebraic subsets of $\mathbb{A}^2$ are $\emptyset, \mathbb{A}^2, \{x\}$ for $x \in \mathbb{A}^1$ and irreducible algebraic plane curves i.e. $V(F)$ for $F \in K[X, Y]$ irreducible.